exercise : compatible operator A an B

Q. $|\alpha\rangle \xrightarrow{\text{A measurement}} |?\rangle \xrightarrow{\text{B measurement}} |?\rangle \xrightarrow{\text{A meas.}} |?\rangle$.

A. i) non degenerate : obvious!

$|\alpha\rangle \xrightarrow{A} |a,b\rangle \xrightarrow{B} |a,b\rangle \xrightarrow{A} |a,b\rangle$.

ii) degenerate (n-fold)

$|\alpha\rangle \xrightarrow{A} \boxed{?} \xrightarrow{B} \boxed{?} \xrightarrow{A} \boxed{?}$

$\sum_{i}^{n} c_a^{(i)} |a, b^{(i)}\rangle$ $\qquad |a, b^{(j)}\rangle \qquad |a, b^{(j)}\rangle$.

A and B do not interfere! ("compatible")

: B between A's does not change the measured result "a". (non-deg.)

(4) Incompatible observables.

if $[A, B] \neq 0$, they do interfere with each other.

there're no simultaneous eigenkets. (in general).

exception : in some case, "subspace" has the same eigenkets.

ex. S-orbital : $|l=0\rangle$.

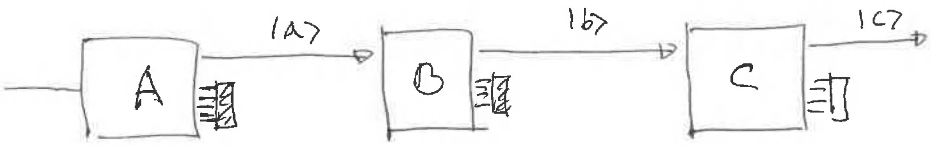
→ this is a simultaneous eigenket of L_x and L_y.

it's isotropic.

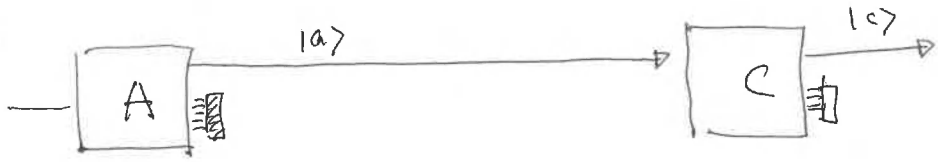
Interference between A and B || $[A,B]\neq 0$

Compare two experiments :

i)



ii)



• What's the difference? - Both gives "$|c\rangle$" !

The Probability to get $|c\rangle$ is ~~different~~ from one to another.

i) $P_1 = |\langle c|b\rangle|^2\,|\langle b|a\rangle|^2$

ii) $P_2 = |\langle c|a\rangle|^2$

☆ What if we "sum up all b" in i) - exp. by repeating exp.

$$\Rightarrow \sum_b |\langle c|b\rangle|^2 |\langle b|a\rangle|^2 = \sum_b \langle c|b\rangle\langle b|a\rangle\langle a|b\rangle\langle b|c\rangle$$

Still, it's different from P_2 !

$$P_2 = |\langle c|a\rangle|^2 = \Big|\sum_{b'} \langle c|b'\rangle\langle b'|a\rangle\Big|^2$$

$$= \sum_{b'b''} \langle c|b'\rangle\langle b'|a\rangle\langle a|b''\rangle\langle b''|c\rangle$$

⇑ This is the "Quantum" nature.

$|a\rangle = |b\rangle = |a,b\rangle$

Note : $P_1 = P_2$ when $[A,B]=0$ or $[B,C]=0$ and "nondegenerate" !

$|b\rangle = |c\rangle = |b,c\rangle$

(5) Example : The Uncertainty Relation (incompatible) (derivation)

def.

$$\Delta A \equiv \overset{\text{Hermitian}}{A} - \langle A \rangle$$: an operator to measure the fluctuation.

dispersion (variance)

$$\langle (\Delta A)^2 \rangle = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle = \underline{\langle A^2 \rangle - \langle A \rangle^2}$$

uncertainty relation :

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2$$

Let's prove this.

proof.

Since $\langle (\Delta A)^2 \rangle = \langle \alpha | \Delta A \cdot \Delta A | \alpha \rangle$, likewise for B

some ket. (arb.)

- The Schwarz inequality gives

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \left| \langle \Delta A \Delta B \rangle \right|^2 .$$

$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$.

(a vector formula : $|\vec{a}|^2 |\vec{b}|^2 \geq |\vec{a} \cdot \vec{b}|$

- $$\Delta A \Delta B = \frac{1}{2} \underbrace{[\Delta A, \Delta B]}_{= [A, B]} + \frac{1}{2} \{ \Delta A, \Delta B \} .$$

$\langle A \rangle$, $\langle B \rangle$ are just numbers.

where $[A, B]$: anti-Hermitian

$$[A, B]^\dagger = (AB - BA)^\dagger = BA - AB = -[A, B] .$$

$\langle [A, B] \rangle$: pure imaginary

$\{ \Delta A, \Delta B \}$: Hermitian.

$\langle \{ \Delta A, \Delta B \} \rangle$: real

$$\langle \Delta A \Delta B \rangle = \bigcirc i + \bigcirc .$$

Thus.

$$\left| \langle \Delta A \Delta B \rangle \right|^2 = \frac{1}{4} \left| \langle [A, B] \rangle \right|^2 + \frac{1}{4} \left| \langle \{ \Delta A, \Delta B \} \rangle \right|^2$$

$$\geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2 .$$

1.5 Change of basis

(1) Transformation operator.

There are multiple choices of base kets to span the same H-space.

$$|\Psi\rangle = \sum_a c_a |a\rangle$$
$$|\Psi\rangle = \sum_b c_b |b\rangle$$
$$|\Psi\rangle = \sum_c c_c |c\rangle \quad \cdots\cdots \text{ many!}$$

$$|a\rangle \xleftrightarrow{U,\,U^\dagger} |b\rangle$$

How ARE they related to each other?

: unitary transformation. //

Theorem.

There exists "U" such that $|b^{(l)}\rangle = U |a^{(l)}\rangle$ $\quad l = 1 \cdots N$ ← dim. of H-space

[orthonormal, completeness]

$\Downarrow$ $\boxed{U : \text{ unitary operator}}$ $\underset{\text{iff.}}{\overset{\text{def.}}{:}}$ $\boxed{\begin{array}{c} U^\dagger U = 1 \\ U U^\dagger = 1 \end{array}}$ ☆

↳ An obvious construction of U :

$$\boxed{U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|.}$$

$\therefore$ $|a\rangle \underset{U^\dagger}{\overset{U}{\rightleftarrows}} |b\rangle$

Check : $\quad U^\dagger U = \sum_{k,l} |a^{(l)}\rangle\underbrace{\langle b^{(l)}|b^{(k)}\rangle}\langle a^{(k)}| = I$

$$U U^\dagger = \sum_{k,l} |b^{(l)}\rangle\langle a^{(l)}|a^{(k)}\rangle\langle b^{(k)}| = I.$$

(2) Transformation Matrix

$$\underset{\text{(old)}}{\{|a\rangle\}} \xrightarrow[\text{: unitary op.}]{U} \underset{\text{(new)}}{\{|b\rangle\}} \qquad U = \sum_k |b^{(k)}\rangle\langle a^{(k)}|$$

- matrix representation of U

$$\langle a^{(k)}|U|a^{(l)}\rangle = \sum_{k'} \langle a^{(k)}|b^{(k')}\rangle \underbrace{\langle a^{(k')}|a^{(l)}\rangle}_{\delta_{k'l}}$$

$$= \langle a^{(k)}|b^{(l)}\rangle$$

$$\equiv U_{kl} \quad \text{: transformation matrix.}$$

- transformation of $|\alpha\rangle$ in the old basis

i.e. $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \to$ coeff. $\langle a'|\alpha\rangle$

to the one in the new basis :

$$|\alpha\rangle = \sum_{b'} |b'\rangle\langle b'|\alpha\rangle \quad \to \text{coeff. } \langle b'|\alpha\rangle.$$

Q.

$$\begin{pmatrix} \langle a'|\alpha\rangle \end{pmatrix} \xrightarrow{?} \begin{pmatrix} \langle b'|\alpha\rangle \end{pmatrix} \quad \text{: matrix representation.}$$

$$\langle b^{(k)}|\alpha\rangle = \sum_l \underbrace{\langle b^{(k)}|a^{(l)}\rangle}_{=U^*_{lk}} \langle a^{(l)}|\alpha\rangle \qquad |b^{(k)}\rangle = U|a^{(k)}\rangle$$

$$= \sum_l \langle a^{(k)}|U^\dagger|a^{(l)}\rangle \langle a^{(l)}|\alpha\rangle$$

※ In matrix representation, $\underset{\text{in a-basis}}{|\alpha\rangle} \xrightarrow{[U^\dagger] \text{ matrix.}} \underset{\text{in b-basis}}{|\alpha\rangle}$.

i.e.

it's a daggar!

$$\begin{pmatrix} \vdots \\ \langle b^{(k)}|\alpha\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & & \\ & (U^\dagger)_{kl} = U^*_{lk} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \langle a^{(l)}|\alpha\rangle \end{pmatrix}$$

not "U"!

Q. What about the matrix rep. of an operator

$$\begin{pmatrix} \ddots & & \\ & \langle a^{(m)} | \cancel{X} | a^{(n)} \rangle & \\ & & \ddots \end{pmatrix} \xrightarrow[|b\rangle = U|a\rangle]{?} \begin{pmatrix} \ddots & & \\ & \langle b^{(k)} | X | b^{(l)} \rangle & \\ & & \ddots \end{pmatrix}$$

$$\langle b^{(k)} | X | b^{(l)} \rangle = \langle b^{(k)} | \underbrace{I}_{\sum_m |a^{(m)}\rangle\langle a^{(m)}|} X \underbrace{I}_{\sum_n |a^{(n)}\rangle\langle a^{(n)}|} | b^{(l)} \rangle$$

$$= \sum_{m,n} \underbrace{\langle b^{(k)} | a^{(m)} \rangle}_{U^\dagger} \langle a^{(m)} | X | a^{(n)} \rangle \underbrace{\langle a^{(n)} | b^{(l)} \rangle}_{U}$$

$$\langle b^{(k)} | a^{(m)} \rangle = U^*_{mk} = (U^\dagger)_{km}$$

$$\Rightarrow \quad X' = U^\dagger X U$$

Similarity transformation.

property. $\operatorname{tr}(X) = \operatorname{tr}(X')$: trace is conserved //

→ an invariant of unitary-matrix transformation.
similarity

It's just a math. property of "trace".

$$\operatorname{tr}(U^\dagger X U) = \operatorname{tr}(X U U^\dagger) = \operatorname{tr}[X]$$

likewise,

$$\operatorname{tr}(|a'\rangle\langle a''|) = \delta_{a'a''}$$

$$\operatorname{tr}(|b'\rangle\langle a'|) = \langle a'|b'\rangle$$

(3) Diagonalization.

- How to solve $B|b^\circ\rangle = b^\circ|b\rangle$? ‖ $|b\rangle$, b > unknowns.

- What do we know? Say, we know some base kets to compute $\langle a_i|B|a_j\rangle$.

$\rightarrow \langle a_i|\cdot(B|b\rangle) = b\langle a_i|b\rangle$

$I = \sum_j |a_j\rangle\langle a_j|$

$$\Rightarrow \sum_j \underbrace{\langle a_i|B|a_j\rangle}_{\text{matrix}} \underbrace{\langle a_j|b\rangle}_{\text{col. vector}} = b \underbrace{\langle a_i|b\rangle}_{\text{col. vector}}.$$

$$\begin{pmatrix} \ddots & & \\ & B_{ij} = \langle a_i|B|a_j\rangle & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \langle a_j|b\rangle \\ \vdots \end{pmatrix} = b \begin{pmatrix} \vdots \\ \langle a_i|b\rangle \\ \vdots \end{pmatrix}.$$

∴ linear equation for the eigenproblem.

$\rightarrow$ to find b : $\det(\tilde{B} - \lambda\tilde{I}) = 0$

and just do what you do for the matrix diagonalization.

The most important thing:

To find relevant (good) base kets to express your operator B. ‖ ex. $S_x = \frac{\hbar}{2}\left[|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right]$ in $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis.

(4) Unitary Equivalent Observables

$$A \xrightarrow[U|a\rangle = |b\rangle]{\text{unitary transforma.}} U A U^{\dagger}, \quad U^{\dagger} = U^{-1}$$

$$\boxed{\text{Same eigenvalues (spectra)}}$$ but different eigenket.

A and $UAU^{\dagger}$ are unitary equivalent observables.

proof

$$A|a^{(l)}\rangle = a^{(l)}|a^{(l)}\rangle$$

$$U A \underbrace{U^{\dagger} U}_{= I} |a^{(l)}\rangle = a^{(l)} U |a^{(l)}\rangle$$

$$\Rightarrow \quad (U A U^{\dagger})|b^{(l)}\rangle = a^{(l)}|b^{(l)}\rangle.$$

unitary transformation does not change spectrum

ex. S_x, S_y, S_z (spin-$\frac{1}{2}$)

These are related through the unitary transformations.

$\rightarrow$ eigenvalues are : $+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$

for all of S_x, S_y, S_z

1.6 Position, Momentum, and Translation

(1) continuous spectra

$\Sigma \to \int$

$\delta_{a_j a_i} \to \delta(\xi_i - \xi_j)$

ex) completeness rel.

$$\int d\xi\, |\xi\rangle\langle\xi| = 1$$

ex) $\langle\xi'|\xi\rangle = \delta(\xi' - \xi)$

ex) $\langle\beta|\alpha\rangle = \int d\xi\, \langle\beta|\xi\rangle\langle\xi|\alpha\rangle$

Notation Note !!!

"tilde" $\tilde{\xi}|\xi\rangle = \xi|\xi\rangle$ (or $\underset{\sim}{\xi}$)

$\tilde{\xi}$: operator, ξ: c-number

$A|\xi\rangle = \xi|\xi\rangle$

A: capital : an operator

$\downarrow$

$A|a\rangle = a|a\rangle$

a: lower cap. : c-number

In Sakurai,

primed symbols : c-number.

unprimed : an operator.

ex. $x|x'\rangle = x'|x'\rangle$

How to read

$$\tilde{\xi}|\xi\rangle = \xi|\xi\rangle :$$

ket $|\xi\rangle$ is an eigenket of operator $\tilde{\xi}$

with eigenvalue ξ.

(2) Position Eigenkets and Position Measurements

- $\tilde{x}|x\rangle = x|x\rangle$

$|x\rangle$ → position eigenket. (localized at x)

x ← position : an Eigenvalue of $\tilde{x}$

$\tilde{x}$: x-operator measuring a position from $|x\rangle$

- completeness relation

$$\int dx\, |x\rangle\langle x| = 1 \quad \Rightarrow \text{definite integral}$$

↳ over all space

- The state ket for an arbitrary physical state

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\alpha\rangle$$

$\leftarrow$ continuum version of $|\alpha\rangle = \sum_a |a\rangle\langle a|\alpha\rangle$

$\downarrow$

probability to find $|\alpha\rangle$ in the narrow interval around x.

$$= |\langle x|\alpha\rangle|^2 dx$$

probability density.

- In 3D, $|\vec{x}\rangle \equiv |x, y, z\rangle$

$$\underbrace{\tilde{x}|\vec{x}\rangle = x|\vec{x}\rangle, \quad \tilde{y}|\vec{x}\rangle = y|\vec{x}\rangle, \quad \tilde{z}|\vec{x}\rangle = z|\vec{x}\rangle}_{\text{"simultaneous" eigenket!}}$$

$\leftarrow [\tilde{x}_i, \tilde{x}_j] = 0$.

(3) Translation operator.

$$|\vec{x}\rangle \xrightarrow[\mathcal{J}(\delta\vec{x})]{} |\vec{x}+\delta\vec{x}\rangle$$

$\mathcal{J}$ make translation from $\vec{x}$ to $\vec{x}+\delta\vec{x}$

"infinitesimal"

$$\mathcal{J}(\delta\vec{x})|\vec{x}\rangle = |\vec{x}+\delta\vec{x}\rangle$$

meaning: $\delta\vec{x}$ is too small to change anything else.